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Fields in the Neighborhood of a Caustic

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Abstract

Starting with Picht's solution of the wave equation in the form of an integral of plane waves over a caustic we derive expressions for the solution which depend entirely on the geometry of the problem. An asymptotic evaluation of Picht's integral for high frequencies gives the desired result which can have a number of different forms depending on the precise region where the field is being observed. In particular, a change in the asymptotic field occurs in going from the regular ray field to the caustic or from a regular part of the caustic to a cusp.

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1. Introduction

In this paper we shall consider a particular class of solutions of the wave equation in two dimensions,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + k^2 n^2 u = 0 \qquad (n = \text{constant}), \tag{1.1}$$

in a form which is especially suited for dealing with the case where the wavelength is asymptotically small. It is well known that in the small wavelength limit of diffraction or propagation phenomena the geometrical features dominate the underlying physical situation. Accordingly, the solutions we shall use are completely determined by geometrical considerations.

The geometry of a field can be described optically by a set of rays which may be regarded as the paths of energy flow. Where the rays converge to a focus, or form an envelope (also known as a caustic), it may be assumed that the energy density or the field amplitude is large. Where there is a region containing no rays at all, the energy does not propagate. Given the ray system, the field amplitude in the small wavelength limit can be computed at most points of space by well known methods. In fact, the optical field amplitude has been calculated even at points on the smooth part of the caustic [1], although the field becomes infinite in the small wavelength limit on a caustic. It is known that the dependence of the asymptotic field on the wavelength changes suddenly along a ray as it passes through a point on a caustic, and a change must occur again in a different way where the caustic has a cusp. We shall investigate the behavior of

the asymptotic field in those singular regions by means of a class of explicitly given fields whose construction is due to J. Picht[2]. The Picht fields are a generalization of one used by Debye[3] for investigating the case of a ray system which converges to a point focus.

If any field whatever has a geometrical optics limit where the ray system is regular, there will exist a superposition of Picht fields which, together, have the same asymptotic limit. Moreover, the Picht field will have the same asymptotic limit on the caustics of the ray system as the given field, a fact which can be easily seen from an application of Green's theorem[4]. Thus, the conclusions we shall reach are actually more general than might be expected at first.

What we desire here is a purely geometrical description of the field, a description which in principle would allow us to find the field amplitudes graphically from a graphic reproduction of the corresponding ray system.

2. The Picht Solution

To construct the exact solution of (1.1) given by Picht, we must first determine the ray system of the corresponding geometricaloptics problem. This can be done whether we are interested in diffraction by an obstacle in a homogeneous medium, in which case the rays are straight lines, or in propagation through an inhomogeneous medium, in which case the rays are curved lines. In any event the ray system will generally form one or more envelopes known as caustics. The caustic

curves, on the other hand, completely determine the ray system, for at each point on a caustic there is a tangent ray, and every ray is tangent to one of the caustics.

The geometrical optics approximation fails at the caustics because there, the true wave amplitude grows indefinitely with the frequency. Our attention will be fixed upon the immediate neighborhood of the caustic. Thus, if the rays of the system are curved, we can replace them by the ray system composed of straight line rays having the same envelopes as the original system; for infinitesimal distances there is no distinction between straight lines and sufficiently smooth curved lines.

Let $\vec{P}$ be the position vector to the point at which the field is to be observed; let $\vec{R}(s)$ be the radius vector to the caustic C, and s the arclength parameter along C. Let $\rho(s)$ be the radius of curvature of C, and let $A(s)$ be a quantity determined by the initial distribution of the field amplitude, e.g. at the source. We have, for the unit tangent vector to C,

$$(2.1)\qquad \frac{d\vec{R}}{ds} = \vec{T}(s) \qquad\qquad (|T|^2 = 1).$$

The solution of Picht, corresponding to the caustic C, is then

$$(2.2)\qquad u(\vec{P}) = \int_C A(s)\left\{\exp ik\left[(\vec{P}-\vec{R}(s))\cdot\vec{T}(s)+s\right]\right\}\frac{ds}{\rho(s)} \quad ,$$

where we have taken the index of refraction n to be one for convenience.

(For Figure 1, see page 19a.)

The function exp ik $\vec{P}\cdot\vec{T}$, where $\vec{T}$ is an arbitrary constant vector of unit amplitude is a plane wave solution of the wave equation (1.1). It follows that the integrand in (2.2) and thus that $u(\vec{P})$ itself is a solution of (1.1). Because of the special form of the integrand in (2.2) $u(\vec{P})$ asymptotically approaches the appropriate geometrical optics field at the point $\vec{P}$ as k becomes large.

3. <u>The Asymptotic Solution</u>

The Picht solution (2.2) is valid everywhere for large values of k insofar as the caustic C determines the field $u(\vec{P})$. In what follows we shall investigate in more detail the asymptotic evaluation of (2.2) for large k so that we may obtain approximate expressions for $u(\vec{P})$, which are simpler and depend more explicitly upon the ray geometry near $\vec{P}$ than upon (2.2). In the discussion we shall distinguish between three kinds of regions in which $\vec{P}$ can be located. The point $\vec{P}$ is in the " lit" region if a real ray tangent to C at a real point passes through $\vec{P}$. If no such ray from C can reach $\vec{P}$, we say that $\vec{P}$ is in the "shadow" region. Finally, $\vec{P}$ can lie on the caustic C itself.

In the lit region we can evaluate the integral in (2.2) by stationary phase without running into complications. The derivative with respect to s of the phase of the integrand in (2.2) is

$$(3.1) \qquad \left[\vec{P} - \vec{R}(s)\right] \cdot \vec{N}(s)/\rho(s) \qquad ,$$

where $\vec{N}(s)$ is the unit normal to the caustic C. Clearly (3.1) will vanish for $s = s_0$ if the vector $\vec{P} - \vec{R}(s_0)$ is parallel to the unit

vector $\vec{T}(s_o)$ tangent to C, i.e., when $\vec{P} - \vec{R}(s_o)$ lies along the ray through $\vec{P}$. Since $\vec{P}$ is in the lit region this is possible for real values of the vectors, and the corresponding real value s_o of the arc length parameter s is the stationary point. Let us assume that $\vec{P}$ is on a ray between the source and the stationary point s_o on the caustic.

The second derivative of the phase or the first derivative of (3.1) at the stationary point s_o is

$$-D(s_o)/\rho^2(s_o) \quad , \tag{3.2}$$

where $D(s_o)$ is the magnitude of the vector $\vec{P} - \vec{R}(s_o)$.

By expanding the phase in a power series about the stationary point s_o, and keeping only terms up to the quadratic, we obtain the approximation.

$$u(\vec{P}) \sim \int_C A(s) \exp\left\{ik\left[D(s_o) + s_o - D(s_o)(s - s_o)^2/2\rho^2(s_o)\right]\right\} ds/\rho(s). \tag{3.3}$$

The important contribution from (3.3) comes from the immediate neighborhood of s_o. Thus, we can take the interval of integration from $-\infty$ to $+\infty$ in s without much effect on the result. Now we define

$$\theta - \theta_o = \int_{s_o}^{s} ds/\rho(s) \sim (s - s_o)/\rho(s_o)$$

so that $ds/\rho(s) = d\theta$, whence

$$(3.4) \qquad u(\vec{P}) \sim \left\{\exp ik\left[D(s_o)+s_o\right]\right\} \int_{-\infty}^{\infty} A(\theta)\exp\left\{-ikD(s_o)(\theta-\theta_o)^2/2\right\} d\theta$$

$$\sim A(\theta_o)\left\{\exp ik\left[D(s_o)+s_o\right]\right\} \int_{-\infty}^{\infty} \exp\left\{-ikD(s_o)\theta^2/2\right\} d\theta .$$

Finally, we obtain from (3.4) (see Appendix 1):

$$(3.5) \qquad u(\vec{P}) \sim \sqrt{2\pi/kD(s_o)}\, A(s_o) \exp\left\{ik\left[D(s_o)+s_o\right] - i\pi/4\right\} .$$

The quantity $D(s_o)$ can be recognized as the radius of curvature of the wavefront through $\vec{P}$ at the ray which is tangent to the caustic at s_o, for the caustic is always the locus of the centers of curvature of any wavefront. The equation of the wavefront through $\vec{P}$ is

$$(3.6) \qquad D(s) + s = D(s_o) + s_o \quad ;$$

here $D(s)$ is the distance along the ray at s from the caustic to the wavefront, i.e., the radius of curvature of the wavefront at the wavefront end of the ray through s.

If we set

$$(3.6) \qquad A(s_o) = \hat{A}(\theta_o)\sqrt{k/2\pi}\exp(i\pi/4) \qquad ,$$

then we have

$$(3.7) \qquad u(\vec{P}) = \hat{A}(\theta_o)\, D^{-1/2}(s_o) \exp\left\{ik\left[D(s_o)+s_o\right]\right\} .$$

The normalized amplitude $\hat{A}(\theta_o)$ will be determined by the angular

distribution of energy at the source. We recognize (3.7) as the geometrical optics solution for the lit region; in particular, note that the amplitude varies inversely as the square root of the radius of curvature of the wavefront.

If we assume that $\vec{P}$ is in the lit region, not between the source and the caustic, but rather in such a position that the stationary point on the caustic lies on a ray between the source and $\vec{P}$, the results at each stage of the calculation are the same except for the sign of the quadratic term of the power series of the phase in (3.3). The effect is accounted for by multiplying (3.5) or (3.7) by $i = \exp(i\pi/2)$. Thus, we have the well known jump in phase of the geometrical optics solution when a ray passes through the caustic.

When the point $\vec{P}$ is in the shadow region of the caustic, there is no real value of s on the caustic C for which the vector $\vec{P} - \vec{R}(s)$ is parallel to the tangent $\vec{T}(s)$; i.e., one cannot construct a real tangent to C which passes through $\vec{P}$. However, if the radius vector $\vec{R}(s)$ of C is an analytic function of s, it can be continued into the complex plane analytically. Then, defining

$$(3.8) \qquad \frac{d\vec{T}(s)}{ds} = \vec{N}(s)/\rho(s)$$

by analytic continuation for complex values of s, in general one can find a complex value s_0 of s such that the phase is stationary, i.e.,

$$(3.9) \qquad \left[\vec{P} - \vec{R}(s_0)\right]\cdot\vec{N}(s_0) = 0 \quad .$$

The contour C must be deformed until it passes through s_o. The power series approximation for the phase up to quadratic terms can again be used, and the resulting integral evaluated as before. The approximation will resemble (3.7) except that, owing to the complex phase, an exponential decay factor will be present. Thus, in general the solution decays exponentially in the shadow region.

The quantities involved in this case are geometrical only in an extended sense by analytic continuation. If the caustic C is an arbitrary curve and $\vec{P}$ is in the neighborhood of C, in the shadow region, then we may obtain an approximate evaluation of (2.2) in terms of real geometrical quantities. This can be done by replacing C by its circle of curvature at the point on C nearest the point $\vec{P}$ and then evaluating (2.2) as if the integration contour were the circle of curvature. The evaluation of (2.2) for the case of a circular caustic will be carried out later.

When $\vec{P}$ is at a regular point s_o of the caustic C, s_o is also a stationary point, for obviously (3.1) vanishes in this case. However, since D_o is zero, (3.2) must vanish also. To obtain the asymptotic evaluation of (2.2) we must, therefore, use the next non-vanishing term of the power series of the phase in (2.2). The third derivative of the phase is

$$(3.10)\quad -\left[\frac{d}{ds}\left(\frac{d\rho}{ds}\Big/(\rho+1/\rho^3\right)\right] (\vec{F}-\vec{R})\cdot\vec{N} + 3\left(\frac{d\rho}{ds}/\rho^3\right)(\vec{P}-\vec{R})\cdot\vec{T} + 1/\rho^2 \ .$$

At s_o we have

$$\vec{P} - \vec{R} = 0 \ ;$$

thus, (3.10) becomes $1/\rho^2$ and

$$(3.11)\quad u(\vec{P}) \sim A(s_o)(\exp iks_o)\int_{-\infty}^{\infty} \exp\left\{ ik(s-s_o)^3/6\rho^2(s_o)\right\} ds/\rho(s) \ .$$

Proceeding as before, we obtain from (3.11)

$$(3.12)\quad u(\vec{P}) \sim A(s_o)(\exp iks_o)\int_{-\infty}^{\infty} \exp\left\{ ik\ \rho(s_o)\ \theta^3/6\right\}\ d\theta$$

or

$$u(\vec{P}) \sim \frac{2\Gamma(\frac{1}{3})\cos \pi/6}{3\left\{k\rho(s_o)/6\right\}^{1/3}} A(s_o) \exp ik\ s_o \qquad \text{(See Appendix 1)}\ .^*$$

If the caustic C is " flat" (i.e., $\rho = \infty$) at the point s_o, so that the third derivative (3.10) vanishes, we must use the next order term in the power series expansion of the phase in (2.2). For the general case in which the first n terms of the power series vanish we use the $(n+1)^{st}$ term, and the procedure is the same as that for the more regular cases just described.

Where the caustic C has a cusp the radius vector $\vec{R}(s)$ has a singularity. It is convenient to assume that the origin of our coordinate system is at the cusp which we can take to be the point $s = 0$ on C. Then the cusp is at $\vec{R}(0) = 0$. We can also assume that the coordinate system is so oriented that the tangent vector $\vec{T}(s)$ has a horizontal limiting position at $s = 0$; i.e., at $s = 0$

$$\vec{T}(0) = (1,0)\ .$$

Geometrically a cusp is a point at which two branchés of the curve C meet. The field should be computed separately for each branch and the total field obtained by superposing the two contributions.

*This result agrees with that on page 160 of [1] .

Thus, we consider now the contribution of a single branch for which s ranges from zero through all positive values.

It is appropriate to assume a singularity in the curvature $1/\rho(s)$ at the cusp and we shall assume that the singularity is algebraic:

$$(3.13) \qquad 1/\rho(s) = \lambda s^{\alpha} + \mu s^{\beta} + \text{ a function of higher order in s.}$$

Here $\beta > \alpha$ and we also assume that $\alpha > -1$, which is necessary in order that $\vec{T}(s)$ exist at $s = 0$.

Now we can write in general for the components of $\vec{R}(s)$:

$$(3.14) \qquad \vec{R}(s) = \left\{ \int_0^s \left[\cos \int_0^{\sigma} dt/\rho(t) \right] d\sigma ,\ \int_0^s \left[\sin \int_0^{\sigma} dt/\rho(t) \right] d\sigma \right\} ,$$

whence

$$(3.15) \qquad \vec{T}(s) = \left\{ \cos \int_0^s dt/\rho(t) ,\ \sin \int_0^s dt/\rho(t) \right\} ,$$

and

$$(3.16) \qquad \frac{d\vec{T}}{ds} = \vec{N}(s)/\rho(s) = \left\{ -\sin \int_0^s dt/\rho(t) ,\ \cos \int_0^s dt/\rho(t) \right\} /\rho(s) .$$

From (3.14) and (3.15), using the power series expansions for the sin and cos functions about zero we obtain

$$(3.17) \qquad \vec{R} = \left\{ s - \left[\lambda^2/2(\alpha+1)^2(2\alpha+3) \right] s^{2\alpha+3} - \dots ,\right. \\ \left. \left[\lambda/(\alpha+1)(\alpha+2) \right] s^{\alpha+2} + \left[\mu/(\beta+1)(\beta+2) \right] s^{\beta+2} + \dots \right\}$$

$$(3.18) \qquad \vec{T} = \left\{ 1 - \tfrac{1}{2} \left[\left\{ \lambda/(\alpha+1) \right\} s^{\alpha+1} + \left\{ \mu/(\beta+1) \right\} s^{\beta+1} \right]^2 + \dots ,\right. \\ \left. \left[\lambda/(\alpha+1) \right] s^{\alpha+1} + \left[\mu/(\beta+1) \right] s^{\beta+1} + \dots \right\} .$$

Writing $\vec{P} = (x,y)$ we then have for the phase

$$(3.19)\quad \left[\vec{P}-\vec{R}\right]\cdot\vec{T}+s = x + \left[y\lambda/(\alpha+1)\right]s^{\alpha+1} + \left[y\mu/(\beta+1)\right]s^{\beta+1}$$

$$-\left[x\lambda^2/2(\alpha+1)^2\right]s^{2\alpha+2} - \left[x\mu^2/2(\beta+1)^2\right]s^{2\beta+2}$$

$$-\left[x\lambda\mu/(\alpha+1)(\beta+1)\right]s^{\alpha+\beta+2} + \left[\lambda^2/(\alpha+2)(2\alpha+3)\right]s^{2\alpha+3} + \dots .$$

When $\vec{P}$ is at the cusp $x = y = 0$, and the phase becomes

$$(3.20)\quad \left[\lambda^2/(\alpha+2)(\alpha+3)\right]s^{2\alpha+3} + \dots .$$

Along a ray $y = 0$, and the phase becomes near the cusp

$$(3.21)\quad x - \left[x\lambda^2/2(\alpha+1)^2\right]s^{2\alpha+2} + \dots .$$

We can also consider various other cases in which we approach the cusp along different paths, e.g., along $x = 0$, and in each case we take from (3.19) only the lowest order term in s. For these other approaches a more precise relationship between α and β must be given to decide which term is actually of lowest order.

In (2.2) we can replace $A(s)$ by $A(0)$ and the phase by (3.20), (3.21) or by a special case of (3.19), depending on the particular limiting case we wish to compute. Consider, for example, the case of $x = y = 0$. From (2.2), (3.20) and Appendix 1 we have :

$$(3.22)\quad u(\vec{P}) \sim A(0)(\operatorname{sgn}\lambda)|\lambda|^{1/(2\alpha+3)}(2\alpha+3)^{-(\alpha+2)/(2\alpha+3)}$$

$$\cdot\, k^{-(\alpha+1)/(2\alpha+3)}(\alpha+2)^{(\alpha+1)/(2\alpha+3)}\,\Gamma\left[(\alpha+1)/(2\alpha+3)\right]$$

$$\cdot\, \exp\left\{i\pi(\alpha+1)/2(2\alpha+3)\right\} .$$

The contribution of the other branch of the cusp in the case of complete symmetry about the horizontal is calculated in the same way. However, we must be careful that the integration is in such a direction that $\vec{T}(s)$ turns so that its angle with the horizontal

$$\theta = \int_0^s ds/\rho(s)$$

increases. On the other hand, if we assume that the second branch is also given for s, going from zero through positive values, it is clear geometrically that $\rho(s)$ must have a sign opposite that of $\rho(s)$ on the first branch. The integration over C for the second branch then must be taken in the direction, -ds. Now to first order in s we have from (3.13)

$$\operatorname{sgn} \lambda = \operatorname{sgn} \rho(s) \quad .$$

Thus, in the case $x = y = 0$ an examination of (3.22) indicates that a sign change occurs for integration in the direction ds, and, therefore, integration in the direction -ds cancels the effect of the sign change exactly. The contribution of the second branch then will be exactly the same as that of the first branch, and the total field at the cusp will be exactly twice the value given by (3.22) with sgn λ equal to +.

It is interesting to consider also the case where $\vec{P}$ is on the ray $y = 0$ through the cusp. From (2.2), (3.21) and Appendix 1.

$$(3.23) \qquad u(\vec{P}) \sim (\operatorname{sgn} \lambda)\Lambda(0)(\pi/2k|x|)^{\frac{1}{2}} \exp\left\{ikx - (i\pi/4)\operatorname{sgn} x\right\} \quad .$$

The value of $u(\vec{P})$ given by the expression (3.23) when sgn λ is + is exactly one-half that given by (3.5) for a point $\vec{P}$ on an ordinary ray in the lit region. This factor of one-half occurs because we are considering just the effect of one branch of the cusp. The other branch of the cusp produces a contribution to $u(\vec{P})$ identical with that given by (3.23). The sum of the two contributions will then result in a field value for $u(\vec{P})$ which is the same as that occurring for an ordinary position of $\vec{P}$ in a lit region.

For the case of $\vec{P}$ on the line, $x = 0$ orthogonal to the ray $y = 0$, we choose the second and third terms on the right side of (3.19) for the phase:

$$(3.24) \qquad \left[y\lambda/(\alpha+1)\right] s^{\alpha+1} + \left[y\mu/(\beta+1)\right] s^{\beta+1} \quad .$$

From (2.2) and (3.24) we have

$$(3.25) \qquad u(\vec{P}) \sim \lambda A(0) \int_0^\infty s^\alpha \exp ik\left\{\left[y\lambda/(\alpha+1)\right]s^{\alpha+1} + \left[y\mu/(\beta+1)\right]s^{\beta+1}\right\} ds$$

$$= \left[\lambda A(0)/(\alpha+1)\right] \int_0^\infty \exp ik \left\{\left[y\lambda/(\alpha+1)\right]\tau + \left[y\mu/(\beta+1)\right]\tau^{(\beta+1)(\alpha+1)}\right\} d\tau.$$

For certain relations between α and β, e.g., for $(\beta+1)/(\alpha+1)$ an integer, (3.25) can be evaluated easily by stationary phase. For the contribution of the other branch of the cusp we multiply the result (3.25) by -1 and replace λ and μ by $-\lambda$ and $-\mu$.

4. An Example: A Circular Caustic and a Point Focus

Let us consider a simple example of a caustic with no cusp. Let C be a circle of radius ρ with its center at the origin. Then C is given by

$$(4.1) \qquad \vec{R}(s) = \left(\rho \cos(s/\rho),\ \rho \sin(s/\rho)\right).$$

Also we have:

$$\vec{T} = \left(-\sin(s/\rho),\ \cos(s/\rho)\right)$$
$$\vec{N} = -\left(\cos(s/\rho),\ \sin(s/\rho)\right),$$

and ρ is the constant radius of curvature of C. The relation (2.2) becomes

$$(4.2) \qquad \begin{aligned} u(\vec{P}) &= \int_C A(s) \exp\left\{ik\left[-x \sin(s/\rho) + y \cos(s/\rho) + s\right]\right\} ds/\rho \\ &= \int_C B(\theta) \exp\left\{ik\left[r \sin(\phi - \theta) + \rho\theta\right]\right\} d\theta, \end{aligned}$$

where $A\left[s(\theta)\right] = B(\theta)$, $r = |\vec{P}|$ and ϕ is the direction angle of $\vec{P}$, and $s = \rho\theta$. The angle θ_o for which $\vec{P} - \vec{R}$ is normal to $\vec{N}$ can be computed quite easily or seen at once geometrically:

$$(4.3) \qquad \theta_o = \phi - \cos^{-1}(\rho/r).$$

The distance D_o is easily seen to be

$$(4.4) \qquad D_o = (r^2-\rho^2)^{\frac{1}{2}}.$$

From (3.5) for a point $\vec{P}$ in the lit region $r > \rho$ we have

$$(4.5) \qquad u(\vec{P}) \sim \left\{2\pi/k(r^2-\rho^2)^{\frac{1}{2}}\right\}^{\frac{1}{2}} B(\theta_o) \exp\left\{ik\left[(r^2-\rho^2)^{\frac{1}{2}} + \rho\left(\phi - \cos^{-1}(\rho/r)\right) - i\pi/4\right]\right\}$$

where we have used $\rho\theta_o = s_o$ and (4.3).

The result (4.5) holds only in the case that a single portion of the wavefront corresponding to C passes through $\vec{P}$. In general, a spiral shaped wavefront winds around C and, as it propagates, one turn of the spiral after another will pass through $\vec{P}$. At each passage the stationary point is such that the corresponding angle θ_o is increased by 2π. This fact is predictable by the observation that the term $\cos^{-1}(\rho/r)$ of (4.3) is ambiguous. Thus, in general, the field at $\vec{P}$ is

$$(4.6)\quad u(\vec{P})\sim\sum_{n=-\infty}^{\infty} B(\theta_o+2\pi n)\left\{2\pi/k(r^2-\rho^2)^{\frac{1}{2}}\right\}^{\frac{1}{2}}\exp\left\{ik\left[(r^2-\rho^2)^{\frac{1}{2}}+\rho(\phi-\cos^{-1}(\rho/r)-2\pi n)\right]\right\}.$$

Of course $B(\theta_o+2\pi n)$, which is given by the source, may vanish for an argument sufficiently large in magnitude and reduce (4.6) to a finite sum.

For $\vec{P}$ on C where $r = \rho$ we have from (3.12)

$$(4.7)\quad u(\vec{P})\sim\left\{2\Gamma(1/3)(\cos\ \pi/6)/3\left[k\rho/6\right]^{\frac{1}{3}}\right\}A(\rho\phi)\exp(ik\rho\phi)$$

for a single turn of the spiral wavefront. In general,

$$(4.8)\quad u(\vec{P})\sim\left\{2\Gamma(1/3)(\cos\ \pi/6)/3\left[k\rho/6\right]^{\frac{1}{3}}\right\}\sum_{n=-\infty}^{\infty}A(\rho\phi+2n\pi\rho)\exp\left[ik(\rho\phi+2\pi n)\right].$$

In the case of a point focus the radius ρ of C becomes zero, and we might consider this as a limiting case of a circular caustic. Setting $\rho = 0$ in (4.2) we obtain

$$(4.9)\quad u(\vec{P})\sim\int_C B(\theta)\exp\left\{ik\left[r\sin(\phi-\theta)\right]\right\}d\theta\ .$$

Now in the limit $r \to 0$ a stationary point θ_0 will lag behind ϕ by $\pi/2$. It is convenient to replace the function $B(\theta)$ in (4.9) by an angular source distribution function which determines the proportion of energy assigned to each ray. Since the direction of a ray through $\vec{P}$ is ϕ, this can be accomplished by a change of variable

$$(4.10) \qquad \theta + \pi/2 = \xi \ .$$

Correspondingly we set

$$(4.11) \qquad F(\xi) = B(\xi - \pi/2) \ .$$

Then (4.9) becomes

$$(4.12) \qquad u(\vec{P}) = \int_{-\pi}^{\pi} F(\theta)\exp\left\{ikr\cos(\phi - \theta)\right\} d\theta \ ,$$

where we have assumed only a single integration about C which implies that $F(\theta)$ is periodic of period 2π. If this were not true we should be forced to extend the integral (4.12) over a larger interval. We can check on the meaning of $F(\theta)$ with respect to our source by evaluating (4.12) directly by stationary phase:

$$(4.13) \qquad u(\vec{P}) \sim (2\pi/kr)^{\frac{1}{2}} F(\phi)\exp\left\{ikr + i\pi/4\right\} \ ,$$

which holds when $\vec{P}$ is not near the focus.

The result (4.12) agrees with the Debye solution of the perfect focus problem [3].

In case ρ is not zero and the point $\vec{P}$ is inside the circular caustic we have

$$(4.14) \qquad r < \rho \ ,$$

and the stationary point given by (4.3) is complex. Since

$$\cos(\phi - \theta_0) = \frac{\rho}{r} \quad ,$$

it is easily found that in the complex case

$$\theta_0 = \phi + i \log\left\{(\rho - \sqrt{\rho^2 - r^2}\ )/r\right\} \quad . \tag{4.15}$$

Then (4.5) becomes in this case:

$$u(\vec{P}) \sim \left[2\pi/k(\rho^2 - r^2)^{\frac{1}{2}}\right]^{\frac{1}{2}} B(\theta_0) \cdot \exp\left\{-k\left[(\rho^2 - r^2)^{\frac{1}{2}} + \rho \log(\rho/r - \sqrt{\rho^2 - r^2}/r)\right] + ik\phi\right\} , \tag{4.16}$$

where $B(\theta_0)$ is the analytic continuation of the source distribution to the complex angle θ_0 given by (4.15).

5. Conclusions

It can be seen from an examination of the various formulas for the field amplitudes given in Sections 3 and 4 that the field is completely specified in the small wavelength limit by the local geometry determined by the caustics of the ray system. For points in the lit region or on a smooth part of a caustic it is necessary to know the local geometry only to the order of the radius of curvature of the caustic. In the neighborhood of a cusp of a caustic the necessary information must include the order of contact of the two caustic branches. In the shadow region an approximate value for the field amplitude near the caustic is again given in terms of the

curvature of the caustic. For points further away in the shadow region the small wavelength limit of the field amplitude is exponentially small, and it can be argued that non-zero values for this case occur only when the wave length is sufficiently long to introduce non-geometrical considerations into the problem. In case a point in the shadow region is near enough to several points of closest approach on a caustic, the field should consist of a super-position of fields determined by (4.16) for each such point of closest approach.

(For Figures 2 and 3, see page 19a.)

Appendix I

For the sake of completeness we consider the evaluation of an integral of the form

$$I = \int_{\wedge}^{\infty} \exp\left\{ikas^{\lambda}\right\} ds , \tag{A.1}$$

where a is a real constant and λ is a real positive constant.

After a change of variable

$$s = (|a|k)^{-\frac{1}{\lambda}} \tau^{\frac{1}{\lambda}} \exp\left\{(\pi/2\lambda)\operatorname{sgn} a\right\} \tag{A.2}$$

in (A.1) we obtain

$$\begin{aligned} I &= \left\{(|a|k)^{-\frac{1}{\lambda}}/_{\lambda}\right\}\left\{\exp\left[(\pi/2\lambda)\operatorname{sgn} a\right]\right\} \int_{0}^{-i\infty} \tau^{\frac{1}{\lambda}-1} \exp(-\tau)\, d\tau \\ &= \left\{(|a|k)^{-\frac{1}{\lambda}}/_{\lambda}\right\}\left\{\exp\left[(\pi/2\lambda)\operatorname{sgn} a\right]\right\} \Gamma\left(\tfrac{1}{\lambda}\right) . \end{aligned} \tag{A.3}$$

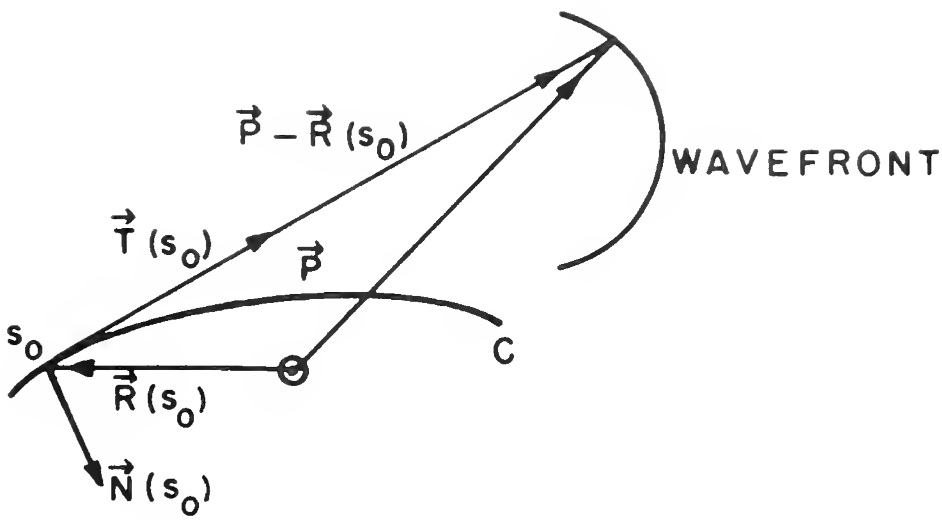

Figure 1.

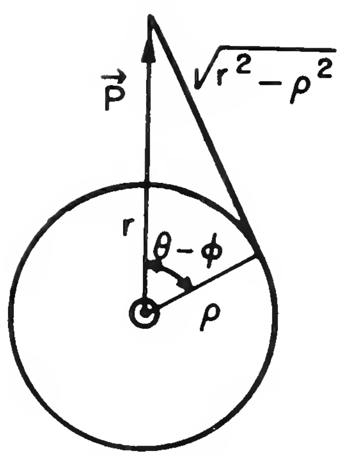

Figure 2.

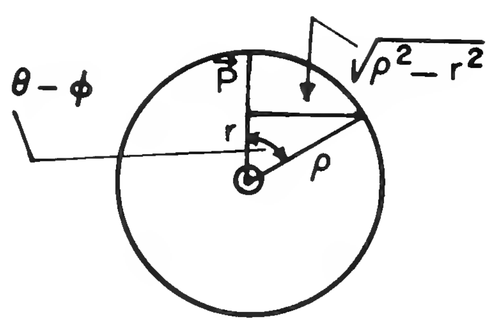

Figure 3.

References

[1] Landau, L. and Lifshitz, E. - The Classical Theory of Fields; Cambridge, Mass., Addison-Wesley Press, Inc., 1951, pp. 155-161.

[2] Picht, J. - Optische Abbildung; Braunschweig, Fried. Vieweg, 1931, Ch.8.

[3] Debye, P. - Das Verhalten von Lichtwellen in der Nahe eines Breunpunktes oder einer Breunlinie; Annalen der Phys., 30, 755 (1909).

[4] I. Kay and J.B. Keller - Asymptotic evaluation of the field at a caustic; J. Appl. Phys.; 25, 876 (1954). Also, N.Y.U., Inst. Math. Sci., Div. EM Res., Research Report No. EM-55, Aug., 1953.

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